Resonance, criticality, and emergence in city traffic investigated in cellular automaton models

A. Varas,[1] M. D. Cornejo,[1] B. A. Toledo,[1,*] V. Muñoz,[1] J. Rogan,[1] R. Zarama,[2] and J. A. Valdivia[1]
[1]*Departamento de Física, Facultad de Ciencias, Universidad de Chile, Santiago, Chile*
[2]*Departamento de Ingeniería Industrial, CeiBA–Complejidad, Universidad de los Andes, Bogotá, Colombia*
(Received 21 August 2009; published 19 November 2009)

The complex behavior that occurs when traffic lights are synchronized is studied for a row of interacting cars. The system is modeled through a cellular automaton. Two strategies are considered: all lights in phase and a "green wave" with a propagating green signal. It is found that the mean velocity near the resonant condition follows a critical scaling law. For the green wave, it is shown that the mean velocity scaling law holds even for random separation between traffic lights and is not dependent on the density. This independence on car density is broken when random perturbations are considered in the car velocity. Random velocity perturbations also have the effect of leading the system to an emergent state, where cars move in clusters, but with an average velocity which is independent of traffic light switching for large injection rates.

DOI: 10.1103/PhysRevE.80.056108 PACS number(s): 89.40.Bb

I. INTRODUCTION

Transportation problems represent an interesting field in physics, mainly due to its high social impact and its emergent properties [1–6]. Ubiquitous among these are the traffic flow and pedestrian flow problems that have been studied extensively in the literature [7–12]. Behaviors such as jamming transitions and chaos have been found to be common [3,13]. In this work we will consider the traffic in a city as represented by a number of interacting cars moving through a sequence of traffic lights, a system which has many nontrivial features [1,2,14–17].

In Ref. [1] it was shown that, for a minimal deterministic model of a car traveling through a sequence of traffic lights, the car dynamics can behave in nontrivial ways, displaying, for example, intermittency and chaos for a realistic range of parameters. The important contribution of that work is that the nontrivial dynamics depends on the finite braking and accelerating capabilities of the cars, which are in general different quantities. The study of the system was characterized for two traffic light phase behaviors: (a) synchronized phases [1] and (b) phases linked by a green wave [2]. The analysis of this model was continued in Ref. [2], which showed that the car dynamics follows a critical scaling law around the resonance condition. Resonance occurs when (a) the traveling time between traffic signals is the same as the period of the signals in the synchronized phase strategy and (b) when the average speed of the car is the same as the green wave velocity in the green wave strategy. These resonances are the boundary between two different dynamics; but, as was shown numerically and analytically for one car, the behavior close to the resonance does not depend on the finite braking and accelerating capabilities of the car. Although the dynamics of a single car has been shown to be very rich [1,2], it is still an idealized situation. The purpose of this paper is to investigate whether the universal behavior close to resonance persists when multiple interacting cars are considered or if it is rather a feature of the simplified single-car model. We will argue that, indeed, the same resonant phenomenon exists when we include interacting cars in the traffic sequence, as it has also been suggested in Ref. [18] for a small and highly correlated cluster. We will study how this resonant behavior changes as we increase the number of interacting cars. For that purpose we will simulate the dynamics of a number of cars by a simple cellular automaton (CA) model. A large number of CA variants have been proposed to simulate city traffic, including many details of the car dynamics. But, for the purpose of this work, we will concentrate on a very simplified CA model since we expect that the critical behavior, close enough to the resonance, should be more or less insensitive to these details (e.g., finite braking and accelerating capabilities, etc. [1,2]).

*btoledo@fisica.ciencias.uchile.cl

II. MODEL

We model the behavior of cars in a sequence of traffic lights with a simple CA. In this model, we divide the distance L_n between successive traffic lights by a number $N_n^c = L_n/\ell$ of cells that can be occupied by a vehicle or can be empty, with ℓ as the size of the cell and n labeling the nth traffic light. The car will move to the next cell in one time step τ if that cell is empty. Conversely, the vehicle will stay in its cell during the next time step if the next cell has a vehicle. Hence, the cars cannot pass each other, and the velocity takes two states of 0 or 1. If the cell is at a traffic light, the car must stay in its cell while the traffic light is red.

Figure 1 shows a possible state of the system, with three cars in the street. Cars always enter the street on the left side and move from left to right. In the lower part of the figure, we represent this state schematically with black (white) cells for occupied (empty) cells. An arrow over a cell boundary represents a traffic light. A cell can only be occupied by a single car or by none at all.

If we assume that (a) the distance between traffic lights is about $L=200$ [m], (b) the size of each cell is about $\ell = 10$ [m], and (c) the cruising velocity is about $v_{\max} = 10$ [m/s] (36 [km/h]), then each time step corresponds to $\tau \simeq 1$ [s]. Let us note that these values are consistent with

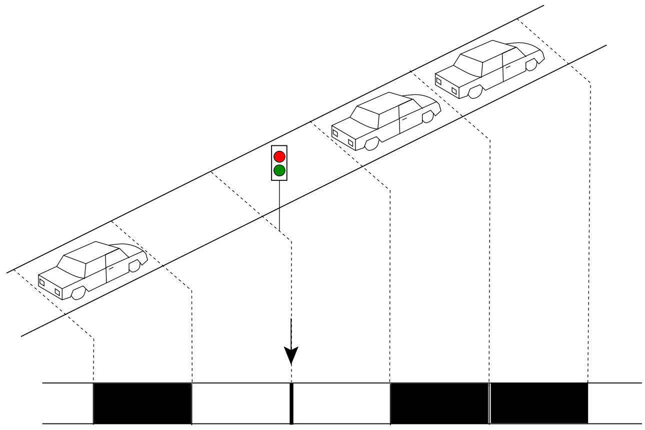

FIG. 1. (Color online) A possible state of the system, with three cars and one traffic light. A schematic representation of the state is shown in the lower part of the figure.

the car having equal accelerating and braking capabilities of $a = v_{\max}^2/2\ell = 5\ [\mathrm{m/s^2}]$.

Switching for the nth traffic light is given by the function $f_n(t) = \sin(\omega_n t + \phi_n)$, where ω_n is the frequency and ϕ_n is the phase delay for that traffic light. Of course, any periodic function can be used. In this work, we consider a single frequency $\omega_n \equiv \omega$ and two traffic light phase behaviors: (a) synchronized phases and (b) phases linked by a green wave [2]. For the first case, in which all traffic lights turn green or red at the same time, a useful parameter is $\bar{\Omega} = T_c/P$, where T_c ($\simeq 20\tau$ for the parameters above) is the minimum travel time between traffic lights and $P = 2\pi/\omega$ is the traffic light period. In these simulations, half of the period P the traffic light is red (meaning the car at the traffic light is at rest) and the other half of the period is green (meaning that the cars can move freely).

In general, each traffic light can have a given time delay Δt_n, with respect to the previous one, so that different traffic lights can turn green and red at different times. For the case of the green wave strategy, a green pulse propagates through the sequence of traffic lights with speed v_{wave}, so that the parameter $\alpha = v_{\max}/v_{\text{wave}}$ will be useful. Then, the phase for the nth traffic light is $\phi_n = -\sum_{m=1}^{n} L_m \omega / v_{\text{wave}}$ [2], so that the time delay between consecutive traffic lights can be written as

$$\Delta t_n = \frac{\phi_{n-1} - \phi_n}{\omega} = \alpha \frac{L_n}{v_{\max}}.$$

In our simulations we use a realistic value of $P = 60\tau = 60$ s.

As we discussed in Sec. I, with this simplified model and these parameters, we expect to represent only the robust dynamics of the critical behavior close to the resonance conditions, where $\bar{\Omega} \sim 1$ and $\alpha \sim 1$, respectively. In order to include some nontrivial effects that may affect the resonance behavior, we will also consider perturbations to the vehicle velocity, by including a parameter r that represents the random probability that a car may stop in the next time step, regardless of the occupation of the next cell. This will be done in Sec. V. We will see that the resonance behavior is

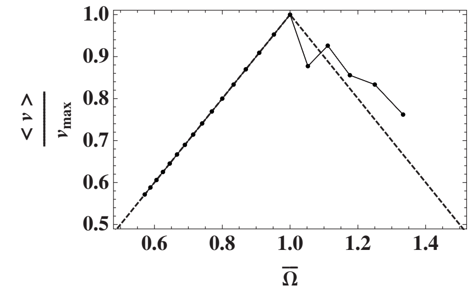

FIG. 2. Average car speed for different traffic signal periods (synchronized traffic lights, $\Delta t_n = 0$; deterministic model, $r = 0$). The dashed line represents the analytical prediction around the critical point $\bar{\Omega} = 1$ [Eq. (1)].

robust against this kind of perturbation, within a certain range of r and car injection rate, even though the detailed dynamics may change considerably.

III. SYNCHRONIZED PHASES

We consider a sequence of $n_s = 50$ traffic lights with $N_n^c = N^c = 20$, hence $T_c = 20\tau$, and consider a set of traffic lights with all phases synchronized to $\Delta t_n = 0$, so that all lights turn green or red at the same time with a period P. Cars enter at the first simulation cell at a rate $1/f$, that is, we try to inject a car every f time steps (if that first cell is occupied, no car is injected). The car density thus depends on f. Also, the model is fully deterministic and cars have $r = 0$ probability of stopping in the next time step, unless the next cell is occupied.

Initially the street is empty. First, we inject cars at the left end at a rate $1/f = 1$, that is, we inject a car at every time step when the first cell is empty. The system is evolved during a time $10^3 P$ to eliminate transient behaviors, and then the dynamics is followed during a time $10^3 P$. We compute the average speed of the cars over the last 30 traffic lights (total distance traveled divided by total travel time). The results are shown in Fig. 2 for the average car speed as a function of the normalized traffic light frequency $\bar{\Omega}$. Let us note that this figure is exactly the same figure as in Ref. [2], where a single car moving through a sequence of traffic lights is considered. In particular, we observe the same critical behavior around $\bar{\Omega} = 1$,

$$\frac{\langle v \rangle}{v_{\max}} \approx 1 - |1 - \bar{\Omega}| \tag{1}$$

for $\bar{\Omega} \approx 1$, which suggests that the behavior around resonance is robust under variations on the car density. This strengthens the proposal that the single-car model in Ref. [1] may model certain aspects of real city traffic. Figure 2 shows that, near resonance, a single car is able to model the dynamics of a cluster of cars.

As we change the injection rate $1/f$ we observe the same critical behavior. In fact the curves for $f = 1, 5, 10, 20$ overlap. The dilute case $f \gtrsim 20$ essentially corresponds to the model in Ref. [2], because there is one car per traffic light on

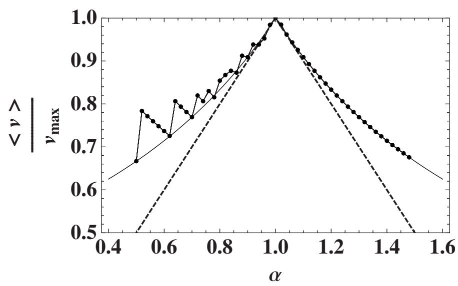

FIG. 3. Average car speed as a function of α, for the green wave strategy, $f=20$, and $r=0$. Dashed lines correspond to the analytical prediction near resonance $\alpha=1$ [Eq. (2)]; solid lines correspond to the more general result (4).

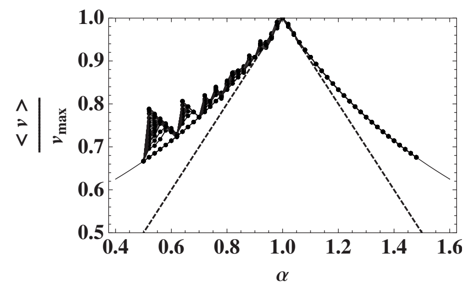

FIG. 4. Same as Fig. 3, but for several values of f, such that $1 \leq f \leq 20$. Linewidth represents the standard deviation of the car velocity over the simulation run.

average. Hence, as a first approximation, the resonant behavior does not depend on the injection rate $1/f$.

IV. GREEN WAVE

Now we consider the green wave strategy, also for the deterministic case $r=0$. That is, the time delays Δt_n for the traffic lights are such that the nth traffic light turns to green a time $\alpha L_n/v_{\max}$ after the $(n-1)$th traffic light. As before we consider $n_s=50$ traffic lights; a fixed street length $N_n^c=N^c=20$, hence $T_c=20\tau$; an empty street initial condition; and an injection rate $1/f$. For the green wave, a larger time is needed to establish a pseudoequilibrium state, so in this case the system is evolved during a time $10^4 P$ in order to drop transient behaviors, and then the dynamics is followed during a time $10^4 P$. The average car speed is calculated across the last 30 traffic lights (total distance traveled divided by total travel time).

Figure 3 shows the results obtained for the average car speed as a function of α for the dilute case $f=20$. The same behavior around resonance is observed as in the single-car model (Fig. 5(a) in Ref. [2]), where the analytical prediction for the average speed is

$$\frac{\langle v \rangle}{v_{\max}} = 1 - |1-\alpha|. \qquad (2)$$

Equation (2) is also plotted in Fig. 3 (dashed line), and it can be noticed that it closely represents simulation data near resonance.

Behavior near resonance also turns out not to depend on car density, or f, as in the synchronized case studied in Sec. III. This is shown in Fig. 4, where results for several values of f from the dilute case $f=20$ to the maximally dense case $f=1$ are plotted. In this and the following figures, the linewidth represents the standard deviation of the car velocity, as calculated when averaging over the simulation run. (This was also considered in Figs. 2 and 3, but the dispersion is negligible in that case.)

Again, a single car is able to model the dynamics of a cluster of cars. We can understand this cluster dynamics by analyzing Fig. 5, which shows snapshots of our simulation for the green wave strategy, and for three values of α: just below, at, and just above resonance.

For the resonant case $\alpha=1.0$ [Fig. 5(b)] cars organize themselves in clusters of length $\ell P/(2\tau)$ containing $P/(4\tau)$ cars [in Fig. 5(b), 15 cars in a cluster of length 30] that never stop. Let us first notice that since cars can only move if the next cell is empty, the minimum distance between consecutive cars is one empty cell. Thus, although $f=1$, cars will quickly organize in a sequence of alternating occupied and empty cells as in Fig. 5(b). If there were no traffic lights, this sequence would never end. However, cars can only pass through a traffic light during one half of the period, $P/2$, which corresponds to $M=P/(2\tau)$ time steps. Since one car moves a single cell in one time step, and since cars are always separated by one empty cell, this means that only $M/2=P/(4\tau)$ cars can pass during a traffic light cycle, thus

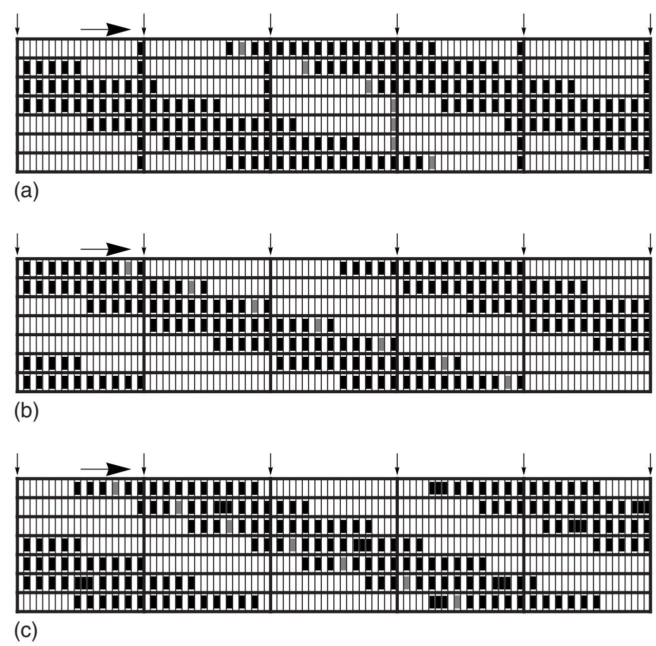

FIG. 5. Dynamics of a car cluster as it propagates in the green wave with (a) $\alpha=0.9$, (b) $\alpha=1.0$, and (c) $\alpha=1.1$. Within each panel, each row represents the state of the simulation from the 20th to the 25th traffic light (see Fig. 1). We let the system evolve for $10^4 P$ iterations, and then we show the same street sector during the times $t=10^4 P+nP/6$, for $n=0-6$. n increases downward in each panel. Horizontal arrows indicate the direction of motion; vertical ones indicate traffic light positions. For the purpose of illustration we have painted gray one of the cars.

breaking the line of cars in clusters of this size, as shown in Fig. 5(b). Since $\alpha=1.0$, $\langle v\rangle/v_{\max}=1$.

For the case $\alpha=1.1$ [Fig. 5(c)], in which case $v_{\max} > v_{\text{wave}}$, a single car starting from a red light would reach the next light before the green wave [2]. Figure 5(c) shows that, when multiple cars are considered, a car cluster is stopped at each traffic light and must wait for the green wave, as revealed by the existence of adjacent occupied cells in Fig. 5(c). Hence, the time traveled between traffic lights is L/v_{wave}, so that we obtain $\langle v\rangle/v_{\max}=1/\alpha$ for the average speed, which is the same result obtained for the single-car model [Eq. (2) for $\alpha>1$]. For the case $\alpha=0.9$ a single car starting from a red light would reach the next light after the green wave [2]. For multiple cars, Fig. 5(a) shows that some of the cars (one in this case) at the tail of the car cluster are stopped at a given traffic light when it turns red, while a car cluster picks the cars left behind by the cluster that is moving ahead. Even though the cluster pulse is moving with speed $v_{\max}$, the individual cars are not capable of staying within a given cluster and are eventually stopped by the traffic lights [notice that at every time step there is a car at each traffic light in Fig. 5(a)]. Hence, the average car speed is $\langle v\rangle/v_{\max} < 1$. As to the car leading the cluster, its dynamics is equivalent to the single-car model. Following Ref. [2], the leading car goes through q traffic lights before being forced to stop, which occurs when

$$\frac{qL}{v_{\max}} - \frac{qL}{v_{\text{wave}}} = \frac{P}{2},$$

that is,

$$q = \frac{Pv_{\max}}{2L}\frac{1}{1-\alpha}.$$

Since the leading car covers a distance qL in a time qT_c and then waits a time $P/2$ for the next green light, its average speed is $\langle v\rangle = qL/(qT_c+P/2)$, which can be written as

$$\frac{\langle v\rangle}{v_{\max}} = \frac{1}{1+(1-\alpha)}. \tag{3}$$

This is the same expression given in Eq. (2) close to 1, for $\alpha<1$. Hence,

$$\frac{\langle v\rangle}{v_{\max}} = \begin{cases} \dfrac{1}{\alpha}, & \alpha \ge 1 \\[2ex] \dfrac{1}{1+(1-\alpha)}, & \alpha \le 1, \end{cases} \tag{4}$$

which is a generalization of Eq. (2).

Let us notice that these results are independent of the injection rate $1/f$ of the cars, because once the cluster is formed the leading car behaves as in the single-car case. This is not true when we include random velocity perturbations (Sec. V).

Let us also note that, for the three cases considered in Fig. 5, the dynamics is periodic with period P (the same state is obtained after a time P). This is in contrast with the nonperiodic chaotic orbits found for the single-car model [1,2]. Such chaotic orbits are not present in the CA model presented here, and this is due to the fact that, in this CA model, cars stop or start instantaneously. It is the finite accelerating and braking capabilities of a car which lead to orbits of period 1, 2, 4, and eventually to chaotic orbits [1,2]. A systematic study of the influence of accelerating and braking capabilities will be presented elsewhere.

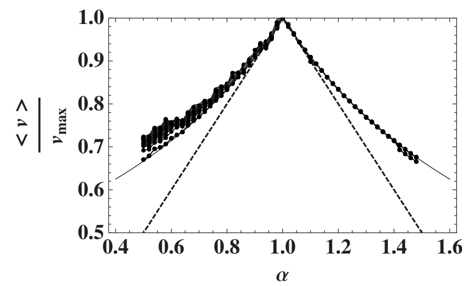

FIG. 6. Same as Fig. 4, but for a random sequence of street lengths. The dashed line represents the analytical prediction of Eq. (2), and the solid line represents the analytical prediction of Eq. (4).

The results presented here do not depend on the street length and do not change if the street length varies randomly. Figure 6 shows the average speed for a sequence of street lengths $L_n = L(1+\beta_n)/2$, where $-0.5 \le \beta_n \le 0.5$ is taken from a uniform distribution and f varies between 1 and 20. We note that critical behavior close to resonance does not change. That is, the critical behavior near $\alpha=1$ found in Ref. [2] is not only universal in the sense that it does not depend on the street length sequence, but also in the sense that it does not depend on the car density, at least within our CA model.

V. RANDOM VELOCITY PERTURBATIONS

In the previous section, it was shown that the critical behavior around resonance is robust under changes in the car density and street length sequence. In this section we will study the effects of random perturbations on car velocity [19–21].

In order to do so, cars in our model will have, at every time step, a probability r of not moving during one time step. Figure 7 shows the average speed for $f=20$ and for different perturbation probabilities r. Unlike the effects studied in the

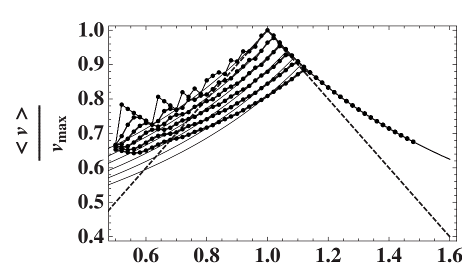

FIG. 7. The average car speed as a function of α for the green wave strategy, for $f=20$. We take equal street lengths $N_n^c = N^c = 20$ and $r=0.0$, 0.02, 0.04, 0.06, 0.08, and 0.10. The larger the value of r, the lower the curve. The solid line corresponds to the result given by Eq. (8), and the dashed line corresponds to that given by Eq. (2).

previous section, resonance and critical behavior are modified if $r \neq 0$. Resonance occurs for a higher value of α and average velocity decreases, with respect to the deterministic ($r=0$) single-car model.

An analytical expression can be obtained to describe the behavior observed in Fig. 7. For any given car, the probability of moving one cell after one time step is $1-r$, the probability of moving one cell after two time steps is $r(1-r)$ (not moving in the first time step, then moving in the second one), etc. The probability of moving one cell after n time steps is $r^{n-1}(1-r)$, in which case its mean velocity is $v_{\max}/n$, and the time it has taken to move is $n\tau$.

The average time to move through a given cell is

$$\langle t \rangle = \sum_{n=1}^{\infty} n r^{n-1}(1-r) = \frac{1}{1-r}. \quad (5)$$

On average the car will advance n cells in a time $n\tau/(1-r)$; therefore, its average speed is $\langle v \rangle / v_{\max} = 1-r$.

Let us now consider the effect of moving through a sequence of traffic lights for one car. Given Eq. (5), the average time to reach the next traffic light is $T_c/(1-r)$. In order to decide whether to continue or stop, this time must be compared with the time L/v_{wave} it takes the green wave to reach the next traffic light. If $T_c/(1-r) \leq L/v_{\text{wave}}$, that is,

$$\alpha \geq \frac{1}{1-r},$$

we will have $\langle v \rangle / v_{\max} = 1/\alpha$.

If the car moves slower than the green wave, $T_c/(1-r) \geq L/v_{\text{wave}}$, the car will go through q traffic lights before being forced to stop. Following a similar argument as that leading to Eq. (3), the car is forced to stop when

$$\frac{qL}{v_{\max}(1-r)} - \frac{qL}{v_{\text{wave}}} = \frac{P}{2}, \quad (6)$$

that is,

$$q = \frac{P v_{\max}}{2L} \frac{1-r}{1-\alpha(1-r)}. \quad (7)$$

Since the car traveled the distance qL during the time $qT_c/(1-r)$, and then waited a time $P/2$ for the next green light, its average speed is $\langle v \rangle = qL(1-r)/[qT_c + (1-r)P/2]$, which can be written as

$$\frac{\langle v \rangle}{v_{\max}} = \frac{1-r}{1+[1-\alpha(1-r)]}.$$

The maximum occurs at $\alpha = 1/(1-r)$, in which case

$$\frac{\langle v \rangle}{v_{\max}} = 1-r.$$

This is equivalent to Eq. (5), which simply means that, in resonance, the traffic light has no effect on car motion.

In summary,

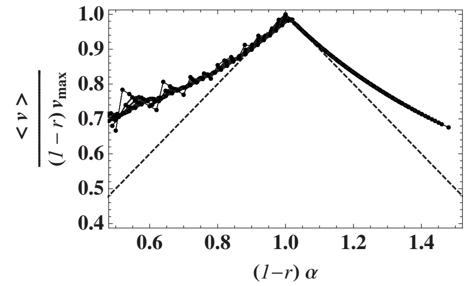

FIG. 8. Same as Fig. 7 near resonance, with average velocities rescaled by $1/(1-r)$ and α rescaled by $(1-r)$. Solid lines correspond to the analytical results (8), and dashed lines correspond to Eq. (2). Linewidth represents the standard deviation of the car velocity over the simulation run.

$$\frac{\langle v \rangle}{v_{\max}} = \begin{cases} \dfrac{1}{\alpha}, & \alpha \geq \dfrac{1}{1-r} \\[2ex] \dfrac{1-r}{1+[1-\alpha(1-r)]}, & \alpha \leq \dfrac{1}{1-r}. \end{cases} \quad (8)$$

Equation (8) suggests that, if average velocities are rescaled by $1/(1-r)$, and α is rescaled by $(1-r)$, then all curves in Fig. 7 should collapse onto a single one for $\alpha \sim 1$. This is indeed the case, as shown in Fig. 8.

We now consider a nondilute case $f=5$. Results are shown in Fig. 9. Comparing with Fig. 7, it is clear that the system is more sensitive to velocity perturbations in the denser case. The behavior near resonance follows only approximately the dilute scaling given by Eq. (8). Thus, it is not independent of the car injection rate, as expected, since the cars that are in front of the cluster affect the motion of the cars behind it. Still the general shape is consistent with the scaling provided by Eq. (8). It is interesting to note that for $r > r_c \sim 0.07$ the average speed becomes independent of α; hence, the traffic lights are no longer relevant in the car dynamics. This critical value r_c turns out to depend on f, as suggested by comparing Figs. 9 and 10. This may be interpreted as a collective state in city traffic, where light switching has no effect on car motion. Against expectations, this state is obtained by increasing random velocity perturbations, even for not very dense situations.

Figure 10 shows the effect of r for the dense case $f=1$. The system is maximally sensitive to velocity perturbations,

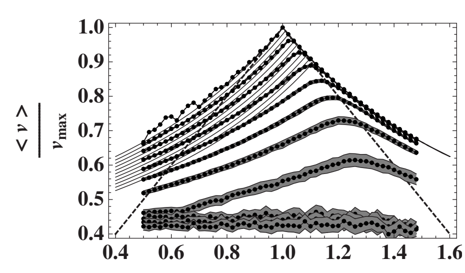

FIG. 9. Same as Fig. 7, but for $f=5$. r varies from 0 to 0.1, in increments of 0.01. Solid lines correspond to the analytical results (8), and dashed lines correspond to Eq. (2). Linewidth represents the standard deviation of the car velocity over the simulation run.

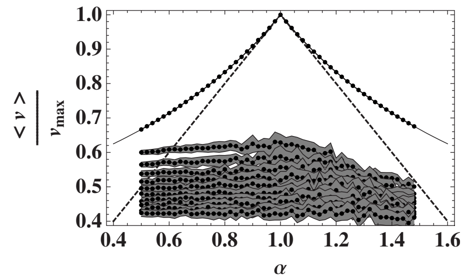

FIG. 10. Same as Fig. 9, but for $f=1$.

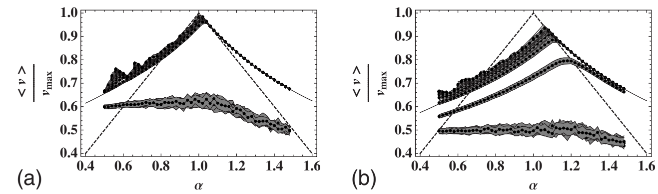

FIG. 12. The average car speed as a function of α for the green wave strategy, for $0 \leq f \leq 20$. The smaller the value of f, the lower the curve. (a) $r=0.01$ and (b) $r=0.05$. In both cases for $f \leq 4$ the average speed is essentially constant as a function of α. Linewidth represents the standard deviation of the car velocity over the simulation run.

so that even for a very small value of r, resonance disappears.

We can visualize the car dynamics in Fig. 11, which is analogous to Fig. 5, but for $r=0.01$. Due to the randomness introduced by the parameter r, the initial conditions (top row in each panel) are not the same in each run. For the same reason, the state evolution is no longer periodic. We can still see the car clusters propagating with a speed close to $v_{\max}$, as individual cars drop out of a given cluster in a random fashion, while cars left behind by the previous cluster are picked up by the next one. This can be seen in Fig. 11, where a particular car has been marked in gray in each panel. In Figs. 11(a) and 11(c), this car belongs to a given cluster; this cluster is broken at a traffic light, and then the gray car will travel with the following cluster once the traffic light turns green. In Fig. 11(b) the situation is similar, except that the gray car is the first one to be stopped by a red traffic light, so that it will be the leading car of the corresponding cluster when the light turns green [see the last row in Fig. 11(b)]. Even though the number of cars being dropped at a given traffic light varies randomly in time, there is a coherent structure (a cluster of cars) propagating in the system, and on average the cars take about the same amount of time to reach the end of the simulation box, independent of v_{wave} (and, hence, of α). There is no noticeable difference among the three panels of Fig. 11, meaning that traffic lights have no effect on car dynamics, as confirmed in Fig. 10.

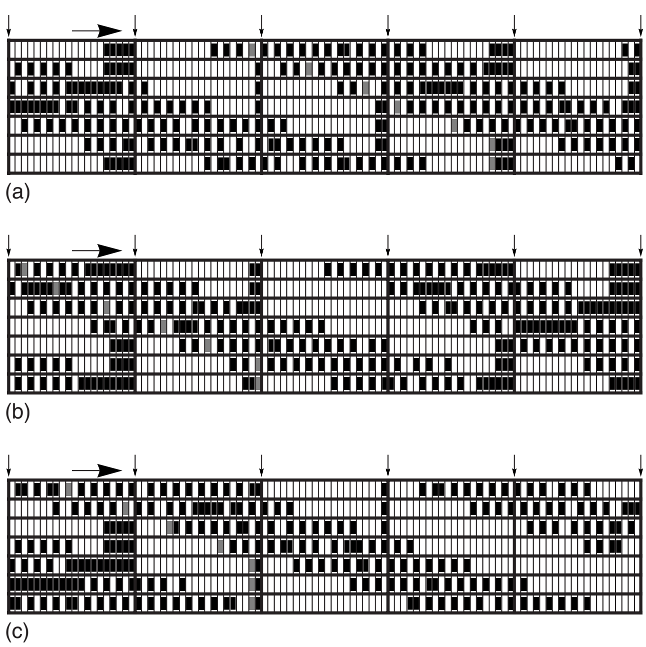

FIG. 11. Same as Fig. 5, but considering the effect of velocity perturbations as given by the parameter r. Injection rate is $1/f=1$ and $r=0.01$. (a) $\alpha=0.9$, (b) $\alpha=1.0$, and (c) $\alpha=1.1$.

The collective state described in the previous paragraph and in Fig. 11 is an emergent state, which would certainly not be possible if traffic lights were not present, because they are responsible for breaking the row of cars in the first place. However, once the clusters are established they move coherently, and the individual car velocity is independent of the traffic light frequency. In a way, this resembles a classical gas subject to binary collisions. Collisions are always present, and they are necessary for the gas to reach equilibrium; but once this is established, collisions simply maintain the equilibrium, but are otherwise not relevant to calculating any thermodynamic variable. In other words, the collisional term in the Boltzmann equation is not zero because there are no collisions, but because collisions manage to make it zero.

Our simulations suggest that the average cluster size $\langle C_s \rangle$ can be estimated—in the absence of noise ($r=0$)—in terms of f only,

$$\langle C_s \rangle \sim \begin{cases} P/(f\tau), & f>4 \\ P/(4\tau), & f \leq 4. \end{cases} \tag{9}$$

For instance, for $P=60\tau$, clusters of 15 cars on average are formed if the injection rate is high enough ($f \leq 4$). For lower injection rates ($f>4$), clusters will have $60/f$ cars on average. Equation (9) states that it is possible to start a cluster of size $\sim P/(4\tau)$ at the left border of the simulation box during the period of the traffic light when $f \leq 4$, as long as r is not large enough to destroy the cluster during one period of the traffic light.

We also calculate the average car speed as a function of α for the same injection rates as in Fig. 4, and for $r=0.01$ [Fig. 12(a)] and $r=0.05$ [Fig. 12(b)]. For $r=0.01$ we have a sharp transition from dilute to collective state at $f \leq 4$, which is exactly when we have the transition $P/4 > P/f$ relevant to Eq. (9). In view of the discussion above, this threshold value $f=4$ is very important because, once we settle to this particular pseudoequilibrium, the incoming flux at the first traffic light ($n=1$) should be equal, on average, to the outgoing flux at the last one ($n=n_s$). (It should be remarked, though, that our simulations show that this threshold value $f=4$ is not independent of the street length L or the traffic light period P.) Let us also note that for $f \leq 4$ the average speed seems to be independent of f and only depends on r as discussed before. For $f > 4$ we note that the resonance is shifted and reduced, but is still there [see Fig. 12(b)]. Hence, a tiny perturbation on car velocities is enough to produce a noticeable change around $\alpha = 1$. We also note that the smaller the size of the cluster that is initiated as the light turns green at the first light, the less effective are the perturbations on the collective motion of the cars. This is expected since perturbations on a given car velocity will have less effect if there are fewer cars behind it.

It is not a coincidence that both the average velocity and the cluster size [see Eq. (9)] do not depend on f, for $f \leq 4$. Both are related to the fact that f is actually the intended injection rate so that, if f is too small, the first light is not able to release, during the light period, more cars than the maximum permitted, e.g., $P/4$ in the case $r=0$.

VI. CONCLUSIONS

We have discussed a CA model for city traffic that includes car interactions and traffic lights and studied the critical behavior close to the resonances $\bar{\Omega} = 1$ (traffic lights in phase) and $\alpha = 1$ (green wave strategy). For a single-car model, this was done in Ref. [2]. The car density is modeled by a parameter f, which measures the frequency with which cars enter the street. Even though some detailed effects such as the finite accelerating and braking capabilities of the cars are missing (which can be improved by allowing multiple velocity states [7], a study we intend to carry out in a future paper), the CA model reproduces results previously obtained in Ref. [2], in the appropriate limit (dilute case $f=20$, with no velocity perturbations $r=0$). Thus, it is shown that the critical behavior found around both resonances is robust not only with respect to the street length sequence [2], but also with respect to the car density.

Also, the fact that this universality holds shows that the single-car model proposed in Ref. [1] is a good model, at least near resonance, for the leading individual in a cluster of cars. However, when cars are allowed to vary their velocities in a random manner (modeled in our case as a probability r of not moving in a given time step), behavior around resonance changes, resonance itself is shifted, and resonant average velocity decreases, depending on the injection rate $1/f$ and r. For a given value of f, there is a threshold value of r, r_c, such that car velocity is independent of α, which can be interpreted as the system entering a collective state, where traffic light switching has no effect on car velocity, with r_c depending on f. Also, for a given value of r, the system enters this collective state if f is small enough.

The situation described above is an emergent state, which is caused by the traffic light switching in the first place; but once it is established, the light switching is not relevant for car dynamics anymore. This resembles a classical gas, where collisions establish and maintain an equilibrium state, but do not otherwise affect macroscopic thermodynamic quantities.

We should point out that, in all cases studied, the initial condition was an empty street. However, the choice of the initial condition is not a trivial matter, and "equilibrium cluster configurations" such as those in Fig. 5 turn out to depend on that choice. We intend to explore this subject in the future.

ACKNOWLEDGMENTS

This project was financially supported by FONDECyT Projects No. 1080658 (V.M.), No. 1071062, No. 1090225 (J.R.), and No. 1070854 (J.A.V).

[1] B. A. Toledo, V. Muñoz, J. Rogan, C. Tenreiro, and J. A. Valdivia, Phys. Rev. E **70**, 016107 (2004).
[2] B. A. Toledo, E. Cerda, J. Rogan, V. Muñoz, C. Tenreiro, R. Zarama, and J. A. Valdivia, Phys. Rev. E **75**, 026108 (2007).
[3] L. A. Wastavino, B. A. Toledo, J. Rogan, R. Zarama, V. Muñoz, and J. A. Valdivia, Physica A **381**, 411 (2007).
[4] T. Nagatani, Physica A **387**, 1637 (2008).
[5] S. Jamison and M. McCartney, Chaos **17**, 033116 (2007).
[6] F. Li, Z.-Y. Gao, and B. Jia, Physica A **385**, 333 (2007).
[7] K. Nagel and M. Schreckenberg, J. Phys. I **2**, 2221 (1992).
[8] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Phys. Rev. E **50**, 822 (1994).
[9] E. Tomer, L. Safonov, and S. Havlin, Phys. Rev. Lett. **84**, 382 (2000).
[10] M. Treiber, A. Hennecke, and D. Helbing, Phys. Rev. E **62**, 1805 (2000).
[11] H. K. Lee, H.-W. Lee, and D. Kim, Phys. Rev. E **64**, 056126 (2001).
[12] A. Varas, M. D. Cornejo, D. Mainemer, B. Toledo, J. Rogan, V. Muñoz, and J. A. Valdivia, Physica A **382**, 631 (2007).
[13] R. Jiang, Q. Wu, and Z. Zhu, Phys. Rev. E **64**, 017101 (2001).
[14] E. Brockfeld, R. Barlovic, A. Schadschneider, and M. Schreckenberg, Phys. Rev. E **64**, 056132 (2001).
[15] M. Sasaki and T. Nagatani, Physica A **325**, 531 (2003).
[16] T. Nagatani, Physica A **347**, 673 (2005).
[17] T. Nagatani, Physica A **377**, 651 (2007).
[18] D. W. Huang and W. N. Huang, Phys. Rev. E **67**, 056124 (2003).
[19] T. Nagatani, Phys. Rev. E **61**, 3534 (2000).
[20] H. Ez-Zahraouy and A. Benyoussef, Eur. Phys. J. B **64**, 573 (2008).
[21] L. C. Davis, Phys. Rev. E **69**, 066110 (2004).